The distribution of Γ

A Fourier decomposing

Jeffery Lewins, January 2014
jl22@cam.ac.uk

Our interest is the magnitude Γ of the difference between the short sides of an irreducible Pythagorean triple. Thus 3,4,5 has $\Gamma = 1$. Triples such as 6,8,10 are rejected as having a common factor and thus reducible with nochange of shape of the triangle. What is the spectrum of this set?

In Dickson's terms, we have $\Gamma = |\Delta_o - \Delta_e|$. Here the odd and even indices are

$\Delta_o = \delta_o^2$ and $\Delta_e = 2\delta^2$. Here δ_o is any odd integer and δ any integer co-prime to δ_o. Dickson's analysis gives $x + \Delta_e = y + \Delta_o = z$ so that Γ is odd.

Table 1 shows starting values of $\Lambda = \Delta_e - \Delta_o$ so that $\Gamma = |\Lambda|$.

Table 1: Λ values as a function of δ / δ_o

20	799	791	X	751	719	679	631	X	511	439
19	721	713	697	673	641	601	553	497	433	X
18	647	X	623	599	X	527	479	X	359	**287**
17	577	569	553	529	497	457	409	353	X	**217**
16	511	503	487	463	431	391	341	287	**223**	**151**
15	449	X	X	401	X	329	281	X	**161**	**89**
14	391	383	367	X	311	271	223	**167**	**103**	**31**
13	337	329	313	289	257	217	X	**113**	**49**	**-23**
12	287	X	263	239	X	167	**119**	**X**	**-1**	**-73**
11	241	233	217	193	161	X	**73**	**17**	**-47**	**-119**
10	199	191	X	151	*119*	**79**	**31**	**X**	**-89**	**-161**
9	161	X	137	113	X	**41**	**-7**	**X**	**-127**	-199
8	127	*119*	103	79	**47**	**7**	**-41**	**-97**	-161	-233
7	97	89	73	X	**17**	**-23**	**-71**	-127	-191	-263
6	71	X	47	**23**	**X**	**-49**	-97	X(133)	-217	-289
5	49	41	X	**-1**	**-31**	-71	*-119*	X	-239	-311
4	31	23	**7**	**-17**	-49	-89	-137	-193	-257	-329
3	17	X	**-7**	-31	X	-103	-151	X	-271	-343
2	7	**-1**	-17	-41	-73	-113	-161	-217	-281	-353
1	*1*	-7	-23	-47	-79	*-119*	-167	-223	-287	-359
δ / δ_o	**1**	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**

X co-factor Bold: recursor italic: special case

Theorem: $\Gamma (\bmod 8) = \pm 1$.

For some δ_o the next odd index is $\delta_o^2 + 4(\delta_o + 1)$. The second term is zero mod(8) since δ_o is odd. For δ_o=1 we have $\Delta_o (\bmod 8) = 1$. Similarly for the even index, if δ is odd $\Delta_e (\bmod 8) = 2 \times \delta_0^2 (\bmod 8) = 2$ and for even integer $\Delta_e (\bmod 8) = 2\delta_e^2 (\bmod 8) = 0$ proving the theorem for the difference of odd and even indices.

Lemma. If p_1, p_2 are primes satisfying $p(mld 8)] \pm 1$ then any product also satisfies. Proof: write $p_1^{m_1} p_2^{m_2} = (8n_1 \pm 1)^{m_1} (8n_2 \pm 1)^{m_2}$ and thus possible members of Γ, where m, n are integers.

Table 1 has no entry for 5 nor any multiple of 5. We can prove this in decimal notation by observing that Δ_o must end in 1, 5 or 9 whilst Δ_e must end in 0, 2 or 8. The only difference yielding 5 or a multiple of 5 is between 0 and 5 and thus reducible. Thus although 25 satisfies the mod8 test, it cannot appear since δ / δ_o have 5 as co-factor and the triple is reducible.

To generalise this, consider any prime p. For any multiples np appearing in δ_o, δ, the triple is reducible so we consider adjacent values $np \pm q$, where q is integer 1, 2… up to (p-1)/2. Thus for p=3, in $\Delta_o = (np \pm 2q)^2 = np(np \pm 4q) + 4q^2$ we have trios 1,3,5 then 7,9,11 and in $\Delta_e = 2(np \pm q)^2 = 2np(np \pm 2q) + 2q^2$ we have -1,0,1 then 2,3;4 then 5,6,*7 etc*. We define reseals over this range for the indices by subtracting the np terms so that we can define *residuals*

$$\operatorname{Re} s(\Delta_e) = 2(q)^2 \text{ and } \operatorname{Res}(\Delta_o) = (2q)^2 = 2\operatorname{Re} s(\Delta_e)$$

It is seen that the even index is its own residual in q counting from the np multiple. Correspondingly, the odd index residual is twice the even residual. Table 2 shows some q-values at low p.

We see that 3 and 5 are not only absent from the set Γ but so are their factors. 7, however, is present as a prime and a factor. Similarly 11, 13 and 119 are barred but 17 and 23 present.

Consequently, 15 and 21 are absent but 49 and 7119 present. Readers may wish to extend the table or find a more general proof.

Table 2. Residuals of indices

$\pm q_o, q_e$	1	2	3	4	5	6	7	8	9	10	11
$\mathrm{Re}\,\Delta_o$ $\mathrm{Re}\,\Delta_e$	4 2	16 8	36 18	64 32	100 50	144 72	196 98	256 128	324 1162	400 200	484 242
mod 3	1 2 :	1 2									
mod 5	4 2	1 3 :	1 3	4 2							
mod *7*	*4* *2*	*2* *1*	*1* *4 :*	*1* *4*	*2* *1*	*4* *2*					
mod 9	4 2	7 8	0 0	1 5	4 2	0 0	7 8	4 2			
mod 11	4 2	5 8	3 7	9 10	1 6 :	1 6	9 10	3 7	5 8	4 2	
mod 13	4 2	3 8	10 5	12 6	9 11	1 7 :	Ditto …				
mod *17*	*4* *2*	*16* *8*	*2* *1*	*13* *15*	*15* *16*	*8* *4*	*9* *13*	*1* *9 :*	*ditto* *…*		
mod 19	4 2	16 8	17 18	7 13	5 12	11 15	6 3	9 14	1 10 :	ditto …	
mod *23*	*4* *2*	*16* *8*	*13* *18*	*18* *9*	*8* *4*	*6* *3*	*12* *6*	*3* *13*	*2* *1*	*9* *16*	*1* *12 :*

Italics: valid factor

Going back to the analysis of 5 in decimal numbers, Table 3 shows the last digit in 2p-numbers for the indices for p=3, 5 and 7. We see that for 3 and 5, no indices match to allow 3 or 5 to be a factor of Γ except reducible p and 2p. For p=7, however, every index has multiple matches.

Table 3. Final digit in 2p arithmetic

p 3	Final Δ_o Digit Δ_e	1 2	3 2	1 0	2	2	0								
5	Final Δ_o Digit Δ_e	1 2	9 8	5 8	9 2	1 0	2	8	2	2	0				
7	Final Δ_o Digit Δ_e	1 2	9 8	11 4	7 4	11 8	1 2	1 0	12	8	2	4	8	2	0

Although I have no rigorous proof, we may reasonably expect that no prime p where $p(\bmod 8) \neq \pm 1$ can be present of itself or as a factor in Γ. If it did, there would be some matching q-cells in the residuals mod p so that some multiple of p would be present. Given the double infinite series in m, n it would seem likely that p itself or some odd power of p would not appear. But this would violate the mod(8) restriction and thus it is reasonable to say there are no matching q-cells for such p.

Fourier decomposition

We can say that the set Γ can be represented as a Fourier decomposition into *fundamental*s consisting of those primes p such that $p(\bmod 8) = \pm 1$ and all possible *harmonics and overtones*, powers and products of these fundamentals.

Somewhat fancifully, we may relate these numbers to the numbers in a musical scale. We need seven notes in say the key of C-major before returning to the octave C. But to play the harmonic minor we need an eighth note, E-flat in the key of C. Disingenuously we say there are 8 distinct notes in our mixed scale. There is nothing new in the harmonics taken modulo 8. In our key of C we have B and D corresponding to $p(\bmod 8) = \pm 1$. B to D sounds a minor third while D up to B is a major sixth. Can we say then that we can hear the "music of the spheres" resounding in our set Γ?

Coda

I am conscious I have not given a rigorous proof and I would be glad to hear of improvements. Perhaps a useful computer exercise would be to extend Table 3 and check the claim. I may mention that If p is odd but fails the mod8 test, then its square passes the test but all odd powers fail. This may lead to a proof but certainly makes the result more likely.

A different approach is Pell recursion and a further computing opportunity. In Dickson's formalism we put

$$\psi = \begin{pmatrix} \delta_o \\ \delta \end{pmatrix}, L = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \psi_{n+1} = L\psi_n .$$

Under this operation Γ is invariant and Λ alternates in sign. Thus in Table 1, we can trace the sequence 1, -1,1 from the origin. The inverse operator $L^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$ retraces the sequence as long as $\delta < \delta_o < 2\delta$ so that a wedge appears in the first quant defined by

$$\delta_o = \delta + 1$$
$$\delta_o + 1 = 2\delta^{\bullet}$$

Any Γ within the wedge has a precursor to the left. Ultimately there is a source cell outside the wedge. The columns and rows are monotonic so, for example, a p-value of 3 would appear within the wedge after $\delta_o = 3$ and appear in an external cell in column $\delta_o = 1 \text{ or } 3$. Since it does not, we have proof that 3 is not a member of the set. This can be computer generalised. We can also show that every source cell (except 1 at the origin) has its counterpart of opposite sign, using rotation operators

$$. R_+ = \begin{pmatrix} -1 & 2 \\ -1 & 1 \end{pmatrix} \text{ and } R_- = \begin{pmatrix} 1 & -2 \\ 1 & -1 \end{pmatrix}.$$