The Lewins Conjecture

An extract from "Shelagh's Triangles: a Pythagorean discourse", Jeffery Lewins 2013.

http://www.shelaghlewins.com/other_stuff/shelaghs_triangles.pdf

We recollect the values found for $\Gamma = |\Lambda = \Delta_o - \Delta_e|$ Table x.

Table 13. Λ values as a function of δ / δ_o

20	799	791	X	751	7719	679	631	X	511	439
19	721	713	697	673	641	601	553	497	433	**X**
18	647	X	623	599	X	527	479	X	359	**2287**
17	577	569	553	529	497	457	409	353	**X**	**217**
16	511	503	487	463	431	391	341	287	**223**	**151**
15	449	X	X	401	X	329	281	X	**161**	**89**
14	391	3183	367	X	311	271	223	167	**103**	**31**
13	337	329	313	289	257	217	X	112	**49**	**-23**
12	287	X	263	239	X	167	**119**	X	**-1**	**-73**
11	241	233	217	193	161	X	**73**	**17**	**-47**	**-119**
10	199	191	X	151	119	**79**	**31**	**X**	**-89**	**-161**
9	161	X	137	113	X	**41**	**-7**	**X**	**-127**	-199
8	127	119	103	**79**	**47**	**7**	**-41**	**-97**	64161 7×23	-122
7	+8	89	73	**X**	**17**	**-23**	23-**71**	-127	-191	-263
6	71	X	**47**	**23**	**X**	**-49**	--97	X	-127	-289 17^2
5	49	41	**X**	**-1**	**-31**	-71	-719 -7×17	X	-239	-311
4	31	23	**7**	**-17**	-49 -7^2	-89	-137	-193	-2557	-329 7×47
3	17	X	**-7**	-31	X	-103	-151	X	-271	-343 7^3
2	7	**-1**	-17	-41	-73	-113	-161 -7×23	-217 -7×31	-281	-353
1	1	-7	-23	-47	-79	-119 -7×17	-167	-223	-284 7×41	-359
$\psi = (\delta, \delta_o)$	**1**	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**

X co-factor Bold: recursor

We see that in general there are starting values in the upper and lower segments $\pm\Lambda$ followed by twin traces in the central wedge. Exceptions include $\Lambda = 1$ followed by a single trace and, for example, 17 and $119 = 17 \times 7$ which have double starts.

We see that in the range studied, the $\Gamma -$ values are limited to those primes p (all odd) satisfying $p (\mathrm{mod}\, 8) = \pm 1$ or products of such primes.

Our conjecture then is in two parts:

1. The set Γ is limited to prime factors and their products satisfying $p(\bmod 8)=\pm 1$. Thus 7x7=49 is valid but not 3x3 even though 9(mod8))=1. Since all products of valid primes satisfy the test we have $\Gamma(\bmod 8)=\pm 1$ if the conjecture is valid.

2. That every possible product of valid primes can indeed be found in the set Γ. That is, we have a set Λ

$$\Lambda = \pm \prod_{\substack{n=1\\,=1}}^{\infty,\infty} p_n^m$$

where p satisfies modulo 8=± 1 and consequently $\Gamma(\bmod 8)=\pm 1$.

I can offer a proof for the first part as follows. Our equation is

$$\Lambda = \Delta_e - \Delta_o = 2\delta^2 - \delta_o^2$$

where δ, δ_o are co-prime integers and δ_o is odd.

Such an equation is the proper subject of Gauss's quadratic reciprocity theorem, a deep theorem at the heart of formal algebra.[1] Witness to this complexity is the claim that there are over 200 proofs of the theorem, including one found on Gauss at his death. Fortunately there is an easier proof.

The odd series $\Delta_o = 1, 9, 25, \ldots$ has a general term $(2n[1)^2$ and hence a stepwise increment $(2n+1)^2 - (2n-1)^2 = 8n$. Thus $\Delta_o(\bmod 8) = 1$. For $\Delta_e(\bmod 8) = 2\delta^2(\bmod 8)$ the odd integers therefore contribute 2 and the even give $2(2n)^2 = 8n^2$ and contribute zero. Thus

$$\Gamma(\bmod 8) = (2 \text{ or } 0)\text{-}1 = \pm 1$$

We have proved that all primes p in the set are limited to $p(\bmod 8) = \pm 1$.We still must show that such an invalid prime cannot be a factor of any member of the set even though, for example, 0^2 satisfies the modulo 8 test.

To do this, we turn to the q-values or residuals modulo p. At any multiple of p the residual is zero but this leads to a false, reducible solution. Between successive false solutions the even index has p-1 values but from the symmetry only half can be distinct: $n^{2p-n2}(mlsp) = (p-m)^2(\bmod p)$ leaving (p-1)/2 for the odd-index. Thus the two fields can indeed be disjoint. Thus for an invalid p=5 the full series gives the field 2,3,3,2 and the odd series gives 1,4. For p=7, valid, we have 2,1,4,4,1,2 and 1,2,4, the first disjoint and the second joint.

But we have shown that if p is an invalid prime it cannot appear in Γ so that the two fields are indeed disjoint and hence we have proved that if $p(\bmod 8) \neq \pm 1$ it cannot appear as a factor in the set.

We have checked the harmonics up to $\Gamma = 49$. All are present.

[1] See for example http://www.math.uga.edu/~pete/thuelemmav3.pdf

Table 14. Harmonics up to R=2500[2]

Γ	Factors	$\psi =$ $(\delta_o,\delta_e$	Γ	Factors	$\psi =$ (δ_o,δ_e)	Γ	Factors	$\psi =$ (δ_o,δ_e)
1	1	1,1	17	17	5,2	23	23	5,1
7	7	1,2	119	17×7	19,6	391	23×17	21,5
49	7^2	9,4	833	17×7^2	33,2	61	23×7	12,2
343	7^3	19,3	0289	17^2	19,6	1127	23×7^2	37,11
2401	7^4	51,10	2023	$17^2\times 7$	45,1	529	23^2	27,10
31	31	7,3	41	41	,2	47		477,1
713	31×23	29,2	1271	41×31	27,7	1927	47×41	45,1
527	31×17	12,1	943	41×23	31,3	1457	47×31	53,26
217	31×7	25,2	697	41×17	27,4	1081	47×23	33,2
1519	31×7^2	39,1	287	41×7	17,1	799	47×17	31,9
661	31^2	33,8	2009	41×7^2	47,10	3329	47×7	23,10
			1681	41^2	59,30	2303	47×7^2	53,5
						2209	47^2	51,4

[2] For prime numbers:
http://primes.utm.edu/lists/small/100000.txt

Γ	Factors	$\psi = (\delta_o,\delta_e$	Γ	Factors	$\psi = (\delta_o,\delta_e)$	Γ	Factors	$\psi = (\delta_o,\delta_e)$
71	71	11,5	73	73	9,2	79	79	9,1
2201	71×31	49,10	2263	73×31	61,27	2449	79×3	57,29
1633	71×23	51,22	1679	73×23	41,1	1817	79×23	45,4
1207	71×17	35,3	1241	73×17	37,8	1343	79×17	55,229
497	71×7	23,4	511	73×7	23,3	553	79×7	25,6
89	89	11,4	97	97	15,8	103	103	11,3
2047	89×23	47,9	2231	97×23	53,17	2369	103×23	49,4
1513	89×17	39,2	16449	97×17	41,4	1751	103×17	43,7
623	89×7	25,1	679	97×7	27,5	721	103×7	27,2
113	113	11,2	127	127	15,7	137	137	13,4
1921	113×17	47,12	2159	127×17	47.5	2329	137×17	49,6
791	113×7	29,5	889	127×7	31,6	959	137×7	2,1
151	151	13,3	167	167	13,1	191	191	17,7
1057	151×7	33,4	1169	167×7	37,10	1337	191×7	37,4
199	199	19,9	223	223	15,1	233	233	25,14
1393	199×7	39.9	1561	223×7	43,12	1631	233×7	41,5
241	241	32,10	257	257	35,22	263	263	19,7
1687	241×7	43,9	1799	257×7	43,5	1841	263×7	43,2
271	271	17,3	281	281	17,2	311	311	19,5
1897	271×7	45,8	1967	281×7	47,11	2177	311×7	53,4
313	313	21,8	337	337	25,12	353	353	21,2
2191	313×7	47,3	2359	337×7	51,11	2471	353×7	53,13

Γ	ψ	Γ	ψ	ψ Γ	ψ	Γ	ψ	Γ	ψ
359	19,1	367	23,9	383	25,11	401	23,8	4009	21,4
431	23,7	433	21,2	439	21,1	449	31,16	457	23,6
463	25,9	479	23,5	487	27,11	503	3+,13	521	23,2
569	31,14	577	33,16	599	29,11	601	27,8	607	25,3
617	25,2	631	27,7	641	29,10	647	35,17	673	31,2
719	31,11	727	27,1	743	29,7	751	33,13	761	31,10
7669	29,6	809	294	823	29,3	839	29,1	857	37,16
863	31,7	881	31,2	887	35,13	911	31,5	919	37,15
929	31,4	937	35,12	953	31,2	967	43,41	977	37,14
983	37,11	991	33,7	1009	39,16	1031	37,13	1033	41,18
1039	33,25	1049	43,20	1063	35,9	1087	33,1	1097	35,8
1103	43,17	1129	39,14	1153	35,6	1193	35,4	1201	37,7
1217	35,2	1223	35,1	1231	41,15	1249	51,26	1279	39,11
1327	47,21	1361	37,2	1367	37,1	1399	43,15	1409	47,20
1423	39,7	1433	49,22	1439	49,11	11447	45,17	1471	39,5
1481	41,10	1487	47,19	1489	39,4	15543	51,23	1553	41,8
1559	51,25	1567	55,27	11601	49,20	1607	43.11	1609	41,6
1657	53,24	1663	44,3	1697	47,16	1721	43,8	1753	49,18
1759	47,15	1777	45,6	1783	45,11	1801	51,20	1823	49,17
1831	43,3	1817	43,4	1871	47,13	1879	51,19	1889	49,16
1913	59,28	1951	49,15	1993	45,4	1999	57,25	2017	45,2
2039	361,29	2063	449,13	2081	47,8	2087	25,8	2089	51,16
2111	47,7	2113	49,12	2129	59,26	2137	47,6	2143	55,21
2153	61,28	2161	53,118	2207	47,1	2239	47,9	2273	49,8
2281	57,22	2287	63,29	2297	53,16	2311	67,33	2351	49,5
2377	55,18	2383	49,3	2393	49,2	2399	49,1	2417	53,14
2423	59,23	2441	67,32	2447	55,17	2473	51,8		*e.o.e*

Thus the conjecture is valid up to R=2500.
We have no rigorous proof for the second part of our conjecture but we see it supported as far as the table goes. A constructive proof would be ideal but an attempt to show every number satisfying the modulo 8 test will fail; 15 has no solution.
A computer program to extend the range might go as follows:

Select the range R, say one million;

Find all valid primes p by the usual sieve algorithm up to $\sqrt{R}$, say 1000;

Extend the table $\psi = (\delta, \delta_o)$ **to** $\delta_o^2 = R$ **;**

Construct all product of the primes p up to R;

Check for the occurrence of each harmonic between vertical columns in the lower wedge from $\delta_o = m$ where m is odd and just satisfies $m^2 > \Gamma$ to $\delta_o = n$ where b odd just satisfies $(n-1)^2 > 2 + \Gamma$.

I would be very pleased to hear of such a program.