Shelagh's Triangles:

A Pythagorean Discourse

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When my daughter Shelagh was young, we explored Pythagoras' Theorem and in particular the paradigm right-angled triangle with sides of length 3,4,5. This triangle is of particular interest because its sides are whole numbers. This has practical implications; for example a surveyor can construct a right angle using a length of rope marked off in sections of length 3,4 and 5 units.

Our 3,4,5 paradigm is the first in an infinite series of integer right-angled triangles with sides of increasing length. The sides of such a triangle form what is called a Pythagorean triple (three positive integers x, y, z such that $x^2 + y^2 = z^2$).

Naturally all such triangles became "Shelagh's Triangles". Now I have five granddaughters who have been exposed to the same joy and I have thought rather more about how to classify and generate such Pythagorean triples, an approach I offer here in the form of an index to sequences of such triples. This leads to a highly accurate estimate of the square root of two. For those unfamiliar with the early, pre-Euclid proof of Pythagoras, I add instructions for making the "magic picture frame".

On further study I find that my approach is essentially that of Leonard Eugene Dickson, an American mathematician working in the 1920's, although my emphasis on the two index parameters allows an application to find the square root of two to higher and higher accuracy. The link is made later to Dickson's method and the methods of Euclid and Plato to find Pythagorean triangles.[1]

Sophie's Triples

A Sophie triple is a Pythagorean triple with no common factors – that is, a set of three irreducible integers that form the sides of a right-angled triangle. We can start to explore the properties of Sophie triples.

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[1] Little is known of Pythagoras himself, living c. 550-450 BC. The standard proof is ct Euclid some centuries later.

Let the three sides of a right-angled triangle[2] be called x, y and z where x,y,z are integer numbers (1,2,3, etc.) as big as you like but not infinite. The hypotenuse (the line opposite the right-angle) is always the third number z but x and y for the other sides can interchange.

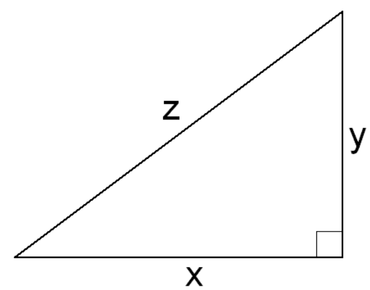

Figure 1: The Right-Angled Triangle with Sides x, y and z

The theorem of Pythagoras says

$$x^2 + y^2 = z^2$$

Young grandchildren will have to have this notation explained of course; see Figure 2 for a visualisation of the squares on the sides of the triangles.

[2] Grandparents may have to explain "right-angle" in terms of four in a symmetric cross.

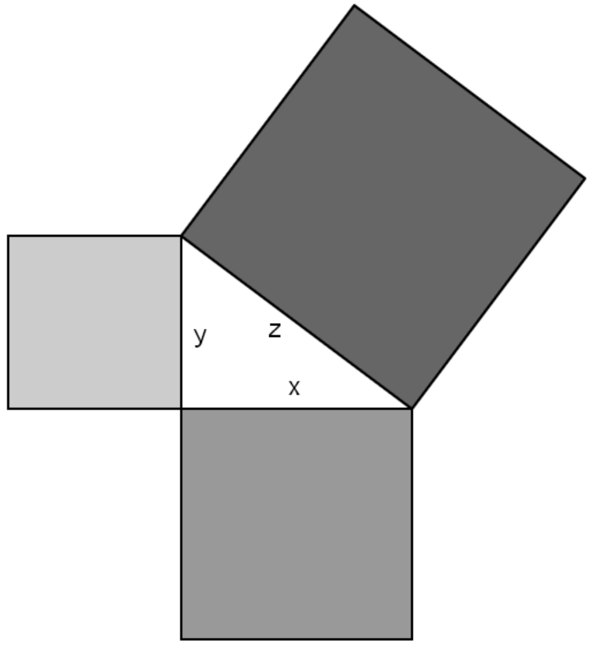

Figure 2: the squares on the sides of the triangle

Our paradigm 3,4,5 then gives us $3^2 + 4^2 = 9 + 16 = 25 = 5^2$ and satisfies Pythagoras, a valid Sophie triple.

Obviously a multiple of a Pythagorean triple, such as 6, 8, 10, is also is a valid Pythagorean triple. But this gives us no new information; the triangle is the same shape and only the unit of length has been scaled by a factor of 2. Only if there is no such factor so that the triple is irreducible, will we call it a Sophie triple. How do we find more Sophie triples?

I introduce the concept of an index Δ for any Sophie triple; the index is the integer number to be added to either x or y to make z. Thus our paradigm (like all triples) has two indices:

$$\Delta_1 = 5 - 4 = 1 \text{ and } \Delta_2 = 5 - 3 = 2$$

Since we have three numbers (the sides of the triangle) linked by one restriction (Pythagoras' Theorem), we would expect to need two independent parameters. Can we prove that Δ_1 and Δ_2 are always different?

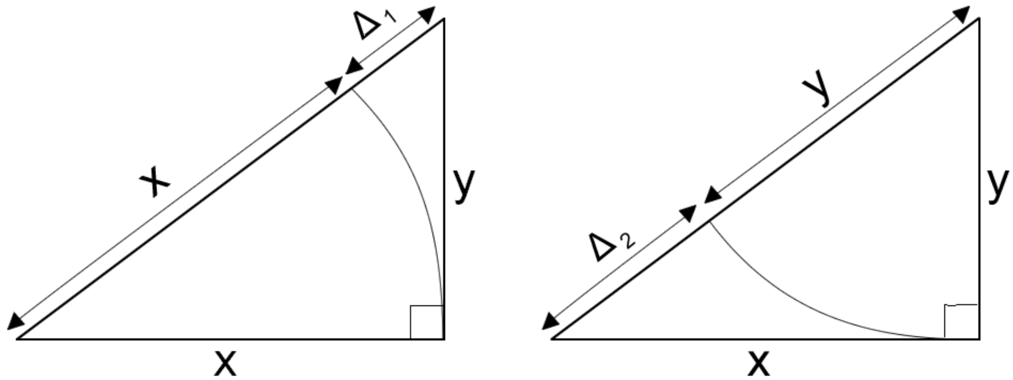

Figure 3: The indices Δ_1 = z - x and Δ_2 = z - y

Theorem 1. Every Sophie triple has two distinct indices

To prove this we assume the fact that the square root of two $\sqrt{2}$ is irrational and cannot be expressed as a ratio of finite integers. If there was but one index we would have $x = y$ and from Pythagoras' theorem $z / x = \sqrt{2}$ which is impossible. Our theorem is proven: since $x \neq y$ a Sophie triple must have two distinct indices, which I will call an Emma pair. The theorem is true for all Pythagorean triples.

Of course the closest we can have x and y is one apart and correspondingly adjacent indices one apart. We call such indices a Charlotte pair and finding such Charlotte pairs is a route to better and better estimates of $\sqrt{2}$.

But first we explore the families of triples using a particular index. We have an *index (or Jeffery) equation*

$$y^2 = z^2 - x^2 = (x + \Delta)^2 - x^2 = 2\Delta x + \Delta^2$$

From this relation, we can search for a perfect square to yield a valid y with fixed index and increasing x.

For example, starting with $\Delta_1 = 1$ and x = 4, we find that y^2 = 2.1.4 + 1 = 9, giving us y = 3 and rediscovering our 3, 4, 5 paradigm.

$\Delta_1 = 2$ and x = 3 gives us y^2 = 2.2.3 + 2.2 = 16, which again yields our 3,4,5 paradigm. This is to be expected because for our 3,4,5 triangle the two indices are 5 – 4 = 1 and 5 – 3 = 2 and we have effectively exchanged Δ_1 and Δ_2.

We can use the index equation to show that certain numbers are not represented in Sophie triples. Recollect that $x \neq y$ and 0 < x,y < z. Suppose y=1 the equation cannot be satisfied: $y^2 = 1 < 2x\Delta + \Delta^2$. Suppose y=2 then we have $y^2 = 4 < 2x\Delta + \Delta^2$ since x>2. We return later to the question of further values absent from Sophie triples.

Theorem 2. A Sophie (irreducible) triple is mixed parity: the three numbers cannot be all odd or all even. Correspondingly the hypotenuse z is odd and the two indices and the sides x, y are of mixed parity, one odd and one even.

We assume some basic arithmetic properties of odd and even numbers, and note that the square of an integer has the same parity as the integer. Thus if x,y are indeed mixed parity then z^2 is the sum of an odd and an even number, hence z^2 is odd, z is odd and the two indices are of mixed parity.

If x, y are both even, then z^2 and z are also even and the triple is reducible so is not a valid Sophie triple – we can rule out this case.

To rule out x,y both odd, we need a lemma.

Lemma. Parity of the Sophie triple

We seek to show that in a Sophie triple (an irreducible Pythagorean triple of positive integers) the hypotenuse is always odd and the other two sides are of mixed parity, one odd and one even. There are three cases: x and y both even, x and y both odd, and the case proposed.

Suppose x and y are both even. Then each square and hence the sum of the squares contains a factor of four. Thus if this is a square number then z is even and the triple is reducible.

Now consider x and y are both odd. First let us examine the remainder of such squares divided by 8, written in modulo notation as n(mod8).

We can write the odd number as 2n - 1, where n is any integer.

$$(2n-1)^2 = 4(n-1)n + 1$$

The leading term has a factor of 8 whether n is odd or even. Thus the remainder modulo 8 of any odd square integer is 1.

The square of any even integer $(2n)^2 = 4n^2$ and has a factor of 4. Thus any even square number modulo 8 is either 0 or 4.

Then if x and y were both odd z would be even. The two odd squares modulo 8 gives 2 but this is incompatible with the even square giving 0 or 4. Hence this case is ruled out.

We have shown therefore that the short sides x and y must be of opposite parity, one odd and one even, and the hypotenuse z is odd, as in our paradigm 3, 4, 5. We return later to the actual and admissible values of x, y and z.

Furthermore, the indices are of mixed parity. Suppose we write $z = x + \Delta_{even}$. Then x is odd and y even; we may have x<y or x>y. From the index equation

$$y^2 = \Delta_r(2x + \Delta_e)$$

with odd x, 2 must appear in odd powers in the factors of the even index.

Corollary. With opposite parity indices the odd index gives y odd, x even so the even index gives y even and x odd in the index equation.

Our paradigms, 3,4,5 and 4,3,5 shows all these features, with indices $\Delta_1 = 1$, $\Delta_2 = 2$ and $\Delta_1 = 2$, $\Delta_2 = 1$.

Each index value corresponds to an infinite series of Sophie triples. Tables 1 and 2 give the first few terms in the sequences corresponding to index = 1 and index = 2. There are some interesting patterns in these tables. For index $\Delta = 1$, y increases in steps of two. The complimentary index is seen to have an odd power of 2 and an even power of odd factors. For index $\Delta = 2$, y increases in steps of four and the complimentary index is the square of odd numbers. Note also that from our index equation, for index = 1, $y^2 = 2x + 1$. Similarly, for index = 2, $y^2 = 4(x + 1)$.

These two indices obviously have one triple in common, because we chose the indices 1, 2 from our paradigm: 3,4,5. A constructive proof follows that every Emma pair of indices leads to a Pythagorean triple. We also have a proof that there are no more than one such triple in common for any two indices.

Table 1

Index $\Delta_1 = z - x = 1$

Sophie Triple (x, y, z) $y^2 = 2x + 1$	**$\Delta_2 = z - y$**
4,3,5	2
12,5,13	8
24,7,25	18
40,9,41	32
60,11,61	50
84,13,85	72
112,15,113	98
144,17,145	128

Table 2

Index $\Delta_1 = z - x = 2$

Sophie Triple (x, y, z) $y^2 = 4(x + 1)$	**$\Delta_2 = z - y$**
3,4,5	1
15,8,17	9
35,12,37	25
63,16,65	49
99,20,101	81
143,24,145	121
195,28,197	169
255,32,257	225

Theorem 3. Two indices have not more than one common triple

Suppose there was another common triple. Let the first common triple be x, y, z, and the second be x', y', z'. Since the indices are common, we can say

$z' - x' = z - x$ and $z' - y' = z - y$

rearranging, we see that

$z' - z = x' - x$ and $z' - z = y' - y$

hence

$z' - z = x' - x = y' - y = n$

Applying Pythagoras' theorem to the seconds triple, we see that $(z+n)^2 = (x+n)^2 + (y+n)^2$ and hence $2zn = 2xn + 2yn + n^2$. For any right-angled triangle, the hypotenuse $z < x + y$ so the equation is invalid for positive n. If $n = 0$ the equation is valid but we have merely recreated the original triple. If n is negative the equation may be satisfied. But then the original triple would be a same-indices triple with a positive n, which we have already refuted.

Other trends seen in Tables 1 and 2 show x/y decreasing with higher terms, the corresponding angle becoming more acute. But we see it is possible to have at the first term, either $x < y$ or $x > y$.

Valid Indices

Rather than study all possible indices and their sequence of triples, it is helpful to find limitations that make indices invalid because they lead to factors that make the triple reducible. The general index equation is

$$y^2 = 2x\Delta + \Delta^2 = \Delta(2x + \Delta)$$

Suppose Δ is even. We can then extract a further factor of two. If the factors of two in Δ are odd then 2Δ will have the form of an odd number times two to an odd power. Then the square root to give integer y must take a further factor of two from x+Δ so that x itself must be even. We would then have z=x+Δ even and a common factor of two making the triple reducible. Thus two must occur in the index only to an odd power.

More generally a valid index may nevertheless have triples with a particular x-value that makes them reducible by virtue of a common factor in x, Δ. Thus not only do we rule out even x in the $\Delta = 2$ sequence but, for example, x=36 in the $\Delta = 9$ sequence where the index equation is

$$y^2 = 9(2x + 9)$$

and would give 27,36,45 reducible to 3,4,5.

Turning to the index itself, consider odd factors in the index. These must occur with even powers or else a perfect square for y^2 would require x must contain a further factor and thus common factor with y and z and hence be reducible.

Table 3 shows the valid indices from 1 to 20.

Table 3. Valid Indices up to 20.

Valid index	Factors	Invalid index	factors
1	1		
2	2		
		3	3
		4	2^2
		5	5
		6	2×3
		7	7
8	2^3		
9	3^2		
		10	2×5
		11	11
		12	$2^2 \times 3$
		13	13
		14	2×7
		15	3×5
		16	2^4
		17	17
18	2×3^2		
		19	19
		20	$2^2 \times 5$

Note that 1 is also a factor of all indices.

Thus valid indices are relatively rare. I leave the reader to validate indices 25, 32, 49, 50 and to note Charlotte pairs of indices: 1,2; 8,9; 49,50; 288,289.

What are the values of x, y that occur in a Sophie triple, an irreducible Pythagorean triple? Values 1 and 2 can be ruled out immediately by observing for $x^2 = z^2 - y^2$ the squares of adjacent integers are at least three apart: $(n+1)^2 - n^2 = 2n+1$. To go further and rule out the lacuna series 2(2n-1), starting therefore at 2, 6, 10, requires deeper study of valid indices and the construction of explicit solutions for Sophie triples.

Theorem 4. A valid odd index has the form $\Delta = \delta^2$ where δ is an odd integer. The hypothesis sharpens a previous result proving x even and says $y = 2n\delta + \Delta$. The index equation shows $y > \Delta$ and gives

$4n^2\Delta + 4n\delta\Delta = 2x\Delta$. Our solutions are then, for valid odd index,

$$
\begin{aligned}
x &= 2n(n+\delta) & x^2 &= 4n^2(n+\delta)^2 \\
y &= 2n\delta + \Delta & y^2 &= 4n^2\Delta + 4n\delta\Delta + \Delta^2 \\
z &= 2n(n+\delta) + \Delta & z^2 &= 4n^2(n+\delta)^2 + 4n(n+\delta)\Delta + \Delta^2
\end{aligned}
$$

so that we have solutions and can reasonably say these are the general solutions. If we call this odd index Δ_1, δ_1 then the complementary even index is given by

$$\Delta_2 = z - y = x + \Delta_1 - y = 2n^2$$

This is independent of Δ_1 so that all valid sequences for an odd index start with complementary index 2. The corresponding series for valid odd indices is

$$\Delta_1 = (2n-1)^2 = \delta_1^2 \text{ and write } \Delta_2 = 2\delta_2^2$$

where the δ are integers and δ_1 odd. Then

$$x + \Delta_1 = z == y + \Delta_2$$

giving

$$
\begin{aligned}
x &= 2\delta_1\delta_2 + \Delta_2 = 2\delta_2(\delta_1 + \delta_2) \\
y &= 2\delta_1\delta_2 + \Delta_1 = \delta_1(\delta_1 + 2\delta_2) \\
z &= 2\delta_1\delta_2 + \Delta_1 + \Delta_2 = (\delta_1 + \delta_2)^2 + \delta_2^2
\end{aligned}
$$

and for a Sophie primitive triple δ_1, δ_2 (an Amelie pair) must have δ_1 odd and must be coprime, with no common factors. This is essentially the method Dickson used.

The methods of Plato and Euclid

Classically we have Euclid's formulae where m and n are integers, m>n:

$$
\begin{array}{ll}
x = m^2 - n^2 & x^2 = m^4 - 2m^2n^2 + n^4 \\
y = 2mn & y^2 = 4m^2n^2 \\
z = m^2 + n^2 & z^2 = m^4 + 2m^2n^2 + n^4 \\
 & = \left(m^2 + n^2\right)^2
\end{array}
$$

The double infinite series gives all possible Pythagorean triples and the restriction to m,n coprime and opposite parity ensures primitive Sophie triples for distinct shapes. We have $\Delta_2 = 2n^2$ and $\Delta_1 = (m-n)^2$.

In contrast, the method of Plato assumes x to be even, 2n say. Then

$$
\begin{array}{ll}
x = 2n & x^2 = 4n^2 \\
y = n^2 - 1 & y^2 = n^4 - 2n^2 + 1 \\
z = n^2 + 1 & z^2 = n^4 + 2n^2 + 1
\end{array}
$$

and alternate terms can be reduced by a factor of two to give only Sophie triples of index 1 and 2.

Corollary. We can now prove that the lacuna series 2(2m+1) does not appear in the general solution for $x = 2n(n+\delta)$ when the index is odd Consider $\delta = 1$ and m integer. There is no solution m for any n of the equation $n^2 + n = 2m + 1$ since the right=hand side is odd but the left-had side is even. More generally consider $n^2 + n\delta = 2m + 1$ and again with odd index no solution exists.

Charlotte pairs of adjacent indices

Sophie triples can give approximations for irrational square roots in the form z/x and z/y. The approximation for $\sqrt{2}$ based on a Charlotte pair of adjacent valid indices is particularly powerful since one triple gives both upper and lower bounds. First we note that the most accurate approach to $\sqrt{2}$ comes when x and y are adjacent, a Rosie pair. A necessary condition is that the triple has a Charlotte pair of indices.

Lemma 3: A unique Rosie pair of adjacent x,y exists for every Charlotte pair of adjacent indices. That is, the sequences corresponding to the two indices have a unique common triple; that this is unique has been shown more generally already. And if a triple with a Rosie pair of adjacent x,y exists in one Charlotte index sequence it must exist in the other Sophie sequence. Again, I know no counter example.

A constructive proof is to order the triple we seek as x<y<z and note $\frac{z}{x} = \frac{x+\Delta_\alpha}{x} > \sqrt{2}$ where Δ_α is the larger of the Charlotte pair. Evaluating $\frac{\Delta_\beta}{\sqrt{2}-1}$ gives us an irrational number greater than the integer x we seek. Similarly using the smaller index Δ_1 a number less than y is given by $\frac{\Delta\sum\beta}{\sqrt{2}-1}$. Thus for the Charlotte pair 1,2 we obtain 2.414 and 4.828. The integer range is therefore 3,4 and thus bracketing the values sought: 3,4. This supposes we already know a good approximation for $\sqrt{2}$. A crude approximation would say it is lower than 1.5 but higher than 1.4. Then for any Charlotte pair the range is limited to $\frac{1}{0.5} = 2 < \frac{2}{0.4} = 4.5$ a span allowing two and only two adjacent Rosie values for a Charlotte pair. This certainly suggests that the hypothesis is true.

The Rosie pair corresponding to a Charlotte pair is the first term in both sequences arising from indices 1 and 2 . For higher Charlotte pairs the common term can appear later. The next table shows the sequences for indices 8 and 9. The lemma also follows from the unique solutions we have given for x,y as functions of the two indices.

Table 4. Sophie Triples for a Charlotte pair 8,9

Index 8	Index 9		Index 49	Index 50
5,12,13	8,15,17		16,63,65	11,60,61
21,20,29	**20,21,29**		36,77,85	39,80,89
45,28,53	*26,27,45		60.91,109	*75,100,125
77,36,85	56,33,65		**120,119,169**	**119,120.169**

* reducible. Common terms **bold**. The reader might explore the sequence for 288 and 289. A further Charlotte pair can be found near 1680. Note that the pairs are alternatively odd-even and even-odd. Is there a proof?

If we have found a Charlotte pair of adjacent indices we may write down explicitly the corresponding pair of adjacent x,y. We have

$$x + \Delta_1 = 2n(n+\delta_1) + \Delta_1 = z = y + \Delta_2 = 2n\delta_1 + \Delta_1 + \Delta_2$$

so that $2n^2 = \Delta_2 = 2\delta_2^2$ and $x = 2\delta_1\delta_2 + \Delta_2, y = 2\delta_1\delta_2 + \Delta_1$ and thus adjacent.

Square root of Two

A Charlotte pair of adjacent indices provides an approximation for $\sqrt{2}$ with the attraction of precise error bounds. If we order the common triple x<y<z then z/x>$\sqrt{2}$>z/y, Thus the paradigm gives the crude approximation that 5/3>$\sqrt{2}$>5/4 with error bound is 5/12=0.4166666 and the arithmetic mean is 1 2/3. Table 5 gives the first few Charlotte and Rosie pairs with increasing accuracy of the approximation .The approximation rapidly gains accuracy as we move to larger Charlotte pairs.

Table 5. Approximations for $\sqrt{2}$

Charlotte Indices	Rosie Triple	Error Width	Arithmetic Mean	Error %
1,2	2,4,5	0.4166	1.45833	3.12
8,9	20,21,29	0.179	1.41547	0.8928
49,50	119,120,169	0.011835	1.41425	0.00263
288,289,	696,697,985	0.00203045	1.414215	0.0001

Thus our final Charlotte pair gives six figure accuracy in estimating the square root of two (1.41421356…) The reader might like to find other approximations for say $\sqrt{3}$ and $\sqrt{5}$. For example, we cannot have y/x=2 in a Sophie triple or else z/x=$\sqrt{5}$. If y/x>2 then z/x is an upper bound and *vice versa.*

Can we prove that successive Sophie triples with adjacent Charlotte pairs of indices can be predicted by use of the ratio $\sqrt{2}+1$ (known in the 'trade' as the silver ratio[3])? Indeed we can but I point out that this supposes we have an adequate approximation for the square root. Thus we go on to discuss the Pell recurrence relation that gives such triples explicitly.

[3] cf the golden ratio of architecture, $(\sqrt{5}+1)/2$ which considered the desirable ratio for width to height of a building façade, and the 'bronze' ratio of modern paper sizes $\sqrt{2}:1$ so that A3 size paper divides equally into A4 with the same aspect ratio.

Charlotte Pairs and Euler's Construction[4]

We have seen that the silver ratio $J = \sqrt{2} + 1$ plays a role in predicting successive Charlotte pairs of adjacent indices. Euler's work on triangular numbers that are also square numbers shows why this should be so and gives explicitly each Charlotte pair.

A triangular number τ_k is the sum of the integers from 1 to n: $\tau k = \sum_1^k n$.

For example, the number of red snooker balls in their triangular frame with five rows is $\tau_5 = 15$. In the Christmas carol, the Twelve Days of Christmas, the singer successively gets τ_n presents daily, from n=1 to n=12 or a total of 364 presents from their true (and presumably rich) love. From the linearity we see that this sum is given by $\tau_k = k(k+1)/2$. The figure shows the first eight triangular numbers. Our interest is in those triangular numbers that are also perfect squares and the figure shows the first two: 1 and 36.

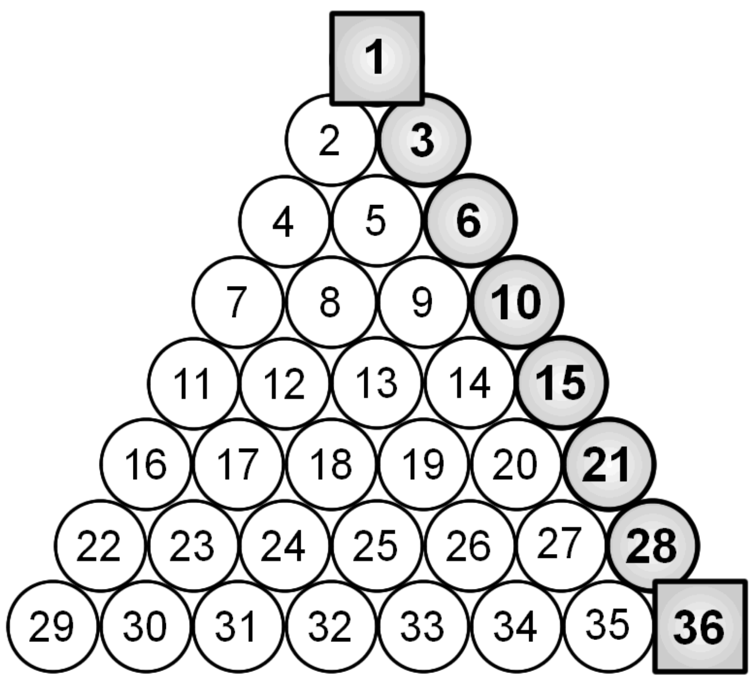

Figure 4: Triangular numbers and the first two common square numbers

That is, we are looking for successive Charlotte pairs that satisfy the index form $\Delta_k(\Delta_k + 1) = 2\delta^2\delta_o^2$ which may be written as

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[4] Leonard Euler 1707-1785.

$$\frac{1}{2}\Delta_k(\Delta_k+1)=\delta^2\delta_o^2$$

On the right is a perfect square of integers while on the left we recognise the triangular number τ_Δ. Euler's formula for the leading index of successive pairs is that

$$\Delta_k=\left(\frac{J^k-J^{*k}}{2}\right)^2$$

Here the complimentary silver ratio $J^*=\sqrt{2}-1$ and consequently $J^*J=1$. It follows that the adjacent index is given by

$$\Delta_k+1=\left(\frac{J^k+J^{*k}}{2}\right)^2$$

Are these indices integer despite the division by 2? At the special case k=0 we have $\Delta_0=1$ with the Charlotte pair 0,1 corresponding to the degenerate Sophie triple 0,1,1. This satisfies Pythagoras but degenerates to a line, useful in the subsequent discussion of Pell numbers – see Figure 5.

At k=1 we have $\Delta_1=\left(\frac{2}{2}\right)^2=1$ with Charlotte pair 1,2 and next $\Delta_2=\left(\frac{4\sqrt{2}}{2}\right)^2=8$ giving indices 8 and 9. We see that not only do alternate terms in the expansion cancel, leaving a doubling of the uncancelled terms and integer index, but we are correctly reproducing the observed Charlotte pairs.

We now have

$$\frac{1}{2}\Delta_k(\Delta_k+1)=\tau_\Delta=\left(\frac{J^{2k}-J^{*2k}}{4\sqrt{2}}\right)^2$$

with successive values k=0: 0, k=1: 2, k=2: 36, etc. corresponding to Charlotte pairs (0,1), (1,2) and (8,9). We have already shown the formula to be integer and thus a square number. But note that the expansion of the silver ratios to even powers leads to cancellation of even terms, leaving odd terms with a common factor of $4\sqrt{2}$ to cancel the denominator.

A related expression for the lower index of order k so that $\tau_k=\frac{1}{2}\Delta_k(\Delta_k+1)$ is

$$\Delta_k = \tfrac{1}{4}\left((3+2\sqrt{2})^k + (3-2\sqrt{2})^k - 2\right)$$

yielding the sequence 1, 8, 49,…

Finally we may see the origin of our observation of the silver ratio (squared) in successive indices and corresponding hypotenuse. We have $J^* < 1$ so that for large k we have $\Delta_k = \left(J^k - J^{*k}\right)^2 \to J^{2k}$ and successive indices increase as J^2.

Strictly speaking we have not proved that there are no other Charlotte pairs not given by the Euler formula but it gives all we have observed.

Pell Numbers and Recurrence Relation

A long established sequence known as Pell numbers provides a slightly different prediction of Sophie Triples and an approximation to the square root of two. The numbers are generated in pairs starting $d_0 = 0, z_0 = 1$ and then computing from

$$d_{n+1} = d_n + 2z_{b+1}, z_{b+1} = z_n + 2d_n$$

Thus we have successively

$$d_1 = 2, z_1 = 5$$
$$d_2 = 12, z_2 = 29$$
$$d_3 = 70, z_3 = 169$$
$$d_4 = 408, z_4 = 985$$
$$d_5 = 2378, z_5 = 5741$$

increasing as the silver ratio J squared.

It is seen that the z-Pell numbers are the hypotenuse of our Sophie triangles. The d-numbers are the Dickson parameter, our term $2\delta_o\delta$ and consequently the difference z-d gives the sum of the indices $2\Delta + 1$. We may therefore readily obtain the complete triple at each stage $x = d + \Delta, y = d + \Delta + 1$ or, alternatively, $y = d + \Delta, x = d + \Delta + 1$.

or $$x, y = \frac{z + d \pm 1}{2}$$

This recurrence relation might be considered more convenient in giving the hypotenuse than the binomial expansion.

The starting numbers for this series, 0,1, correspond to the zeroth or degenerate Sophie triple of 0,1,1 satisfying Pythagoras with two right angles of course.

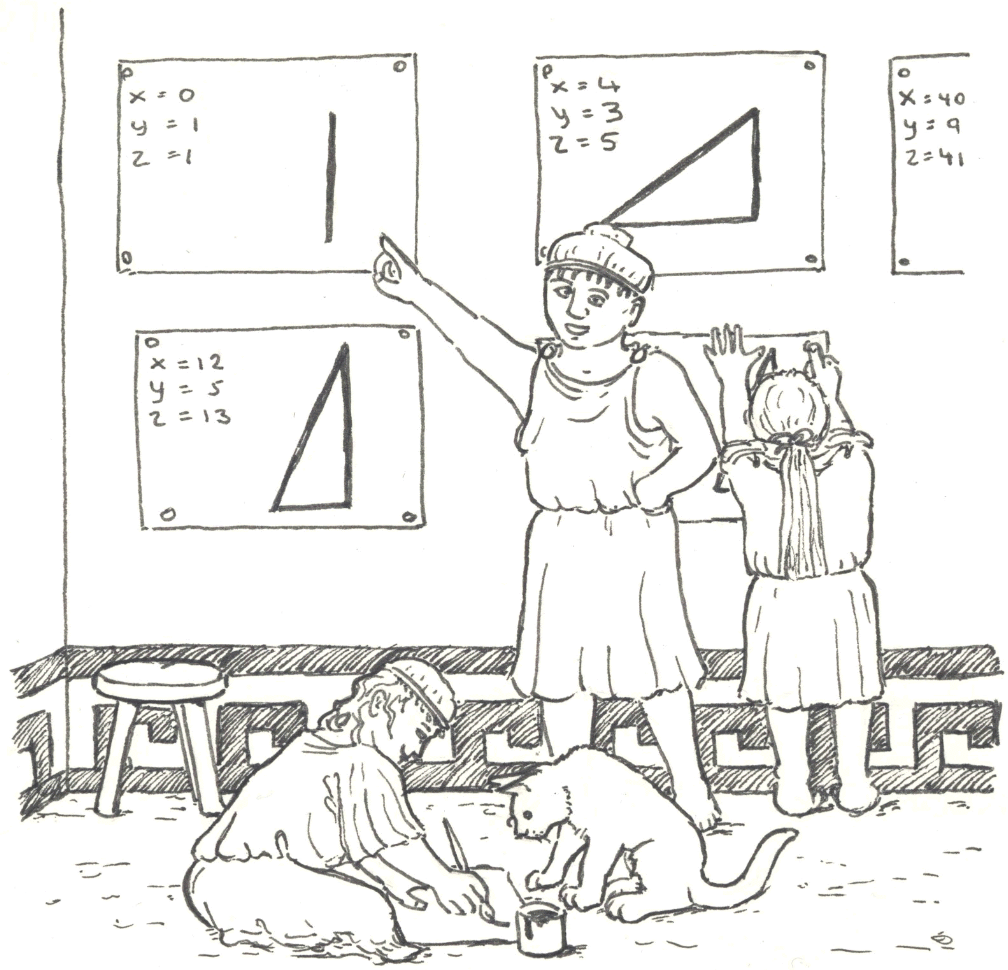

Figure 5: The degenerate triangle

Charlotte: "It doesn't look like a triangle to me, Grandpa."

Pythagoras: "Ah, but 0,1,1 satisfies Pythagoras

and is really the first Sophie Triple."

An explicit expression for x,y comes with a change of notation using the single recurrence relation

$P_0 = 0, P_1 = 1,\ else\ P_n = 2P_{n-1} + P_{b-2}$

Thus we have 0,1 2,5,12,29,70,169,… Then the Sophie triples are given by

$$x = 2P_n P_{n+1}$$
$$y = P_{n+1}^2 - P_n^2$$

so that from Pythagoras

$$z = P_{2n+1} = P_{n+1}^2 + P_n^2$$

and $$d = P_{2n} = (P_{n+1}^2 - P_{n-1}^2)/2$$

Thus for n=1 we have 4,3,5, our first Sophie triangle. For n=2 we have 20,21,29, m=3 gives 120,119,169, etc. Of course the proof of this convenient alternative approach turms on the same binomial expansion of the silver ratio. Substitution shows that after two applications of the recurrence relation $P_{n+2} = 5P_n + 2P_{n-1}$ consistent with the J^2 operation 5.828…

In principle, the binomial expansion allows us to find a high-order triple directly without going through every lower stage of the Pell recurrence relations but modern computers diminish that advantage. Is this the best approximation route to $\sqrt{2}$? Perhaps Newton's more general algorithm is as good as any.

Seen from the perspective of a negative Dickson parameter D<0, we see the four pairings $\pm x \pm y$ yield two values of Γ, corresponding to right- and left-handed triangles. Thus our paradigm triple 3,4,5 yields $\Gamma = 1 \text{ and } 7$.

Admissible triples

From the Dickson analysis we may find admissible values for x, y, z. For the odd side

$$D + \Delta_o = \delta_o(2\delta + \delta_o).$$

Thus admissible values are given by 2n+1: (3,4,5);(5,12,130)), (7,224,25), (9,40,41)…

For the even sided

$$D + \Delta_e = 2\delta(\delta + \delta_o).$$

This has a factor of 4. Values are given by n: (3,4,5), (8,115,17), (5,12,13)…

Then the sum of odd and hypotenuse is given by

$$2D + 2\Delta_o + \Delta_e = 2(\delta + \delta_o)^2$$

and thus by $2n^2$ if we include the degenerate case : (0,1,1), (3,4.5). (5,12,13), (8,15,17)…

Then the sum of the even side and hypotenuse is given by

$2D + 2\Delta_o + \Delta_e = 2(\delta_o + \delta_e)^2$.

These last two results are in the form of even and odd indices, which, with the next, will be relevant studying negative sides.

Finally whereas the difference of odd and even sides is in Γ, the sum is

$2D + \Delta_l + \Delta_e = (2\delta + \delta_o)^2 - \Delta_o$

and this again is in the form of a $\Gamma-$value.

We illustrate this with our paradigm in the table.

Table 6. Reversed sides

x y, z	ψ	D	Δ_e	Δ_o	Λ	Γ
3,4,5	1,1	2	2	11	1	
-3,4,5	1,2	-4	8	1	7	
3,-4,5	3,1	-6	2	9	-7	7
-3,-4,5	3,2	-12	8	9	-1	1

Note that the Dickson factor is mutative if either or both sides are taken negative. We will see the origin of this under Pell precursors. And we note that if both sides reverse Γ is unchanged.

Pell Precursors

Pell numbers can readily be traced backwards and if taken far enough lead to negative values for x,y. The question is then whether these are different triples? Clearly a change of sign for x, y, z does not change Pythagoras but –x, y, x is a reflection rather than a rotation and giving a different triangle, $rt-handed \leftrightarrow left-handed$, we might expect a different Γ.

The table shows four successive precursions.

Table 7. Pell precursors at $\Gamma = 1$

D,z	-12, 5	-2, 1	0, 1	2, 5
Δ_o, Δ_e	8, 9	1, 2	0, 1,	1, 2
x, y ,z	-3, -4, 5	-1, 0, 1	0, 1, 1	3, 4, 5

We have the degenerate triangle as before. All the triples satisfy Pythagoras but is -3,-4,5 a different triangle to 3,4,5? I would say no since the difference can be seen as a rotation through Π by the observer. Correspondingly they have a common $\Gamma = 1$.

Table 8. Pell precursion at $\Gamma = 7$

D, z	-234, 97	-40 , 17	-6 , 5	4, 13
Δ_o, Δ_e	162, 169	25, 32	2, 9	1, 8
x, y, s	-65, -72, 97	-8, -15, 17	3, -4, 5	5, 12, 13
D, z	-176, 73	-30, 13	-4, 5	6, 17
Δ_o, Δ_e	121, 128	18, 25	1, 8	2, 9
x, y, z	-55, -48, 73	-5, -12, 13	-3, 4, 5	8, 15, 17

There are two curiosities in Table B. We have triples -3, 4, 5 and 3, -4, 5 but I argue these are not the same triangles as 3, 4, 5 since they come from reflection not rotation and are thus distinct. Then in the Dickson quadrant (where all four quadrants are the same) $\Gamma = 7$ has two recursion traces, connected by the rotation operator. But with Pell these sequences cross over.

Table 9. Pell precursors at $\Gamma = 17$

D, z	-736, 305	-126, 53	-20, 13	6, 25
Δ_o, Δ_e	512, 529	81, 98	8, 25	1, 18
x, y, z	-207, -224, 305	-28, -45, 53	-5, 12, 13	7, 24, 25
D, z	-330,137	-56,25	-6,13	20,53
Δ_o, Δ_e	225,242	32,49	1,18	8,25
x y, z	-105,-88,137	-24,-7,25	-5,12,13	28,45,53

Again, -5, 12; 13 is not the same triangle as 5; 12, 13 and has a different Γ. Both crossover triples have related Γ values 7.

Table 10. Pell precursors at $\Gamma = 119$

D, z	**-6432, 26665**	**-1102, 461**	**-180, 101**	**22, 145**
Δ_o, Δ_e	**4489, 44608**	**722, 841**	**81, 200**	**2, 121**
X, y, z	**-1943, -18242665**	**-261, -380, 461**	**-99; 20, 101**	**24, 143, 145**
D, z	**-2460, 1021**	**-418, 185**	**-48, 89**	**130, 3349**
Δ_o, Δ_e	**1800, 1681**	**242,361**	**9, 128**	**50, 169**
X, y, z	**-660, -779, 1021**	**-176, -57, 185**	**-39, -80, 89**	**180, 299, 349**
D, z	**-4838, 2005**	**-828, 349**	**-130, 89**	**48, 185**
Δ_o, Δ_e	**3481, 3362**	**529, 648**	**50, 169**	**9, 128**
x, y,	**-1357, -1476, 2005**	**-299, -180, 349**	**-80, 39, 89**	**57, 176, 185**
D, z	**-1850, 769**	**-312, 145**	**-22, 101**	**180, 461**
Δ_o, Δ_e	**1250, 1369**	**169, 288**	**2, 121**	**200, 81**
X, y, z	**-600, -481, 769**	**-141, -24, 145**	**-20, 99, 101**	**261, 380, 461**

We see in Table 10 that Pell precursion links the two rotation related chains at 119 in the Dickson quadrant. At the cross-over with only one of the triple negative, we have a triangle that with positive sides would have $\Gamma = 479$, 79 and 41.

Pell precursion then yields negative values for one or both of x and y. Our earlier results for the sum of two sides the change in indices on reversing the relevant sides. The change in x+y gives the new Γ but not the indices since this is multi-valued.

Pell Amplitude Convergence

For the sequence of Sophie triples having $\Gamma = 1$ we showed that successive values of the hypotenuse z increase with an amplitude ratio that approaches the square of the silver ratio J, using Euler's formula for Pascal's triangular numbers. Table 1 shows the first few Pell numbers starting from the degenerate Sophie triple (0,1,1) reproducing the sequence for $\Gamma = 1$ and showing convergence towards J=2.414….

Table 11. Discrepancy amplitude ratios

P_{n+1}/P_n	1/0	2/1	5/2	12/5	29/12	70/29
Value	∞	2	2.5	2.4	2.417…	2.41379…

That is, for $\Gamma = 1$, the lower index is given explicitly by

$$\Delta_k = \left(\frac{J^k - J^{*k}}{2} \right)^2 .$$

With increasing k terms in J dominate.

This result may be generalised for any Γ using Pell recursion with successive Pell numbers having an amplitude ratio that converges toward J.

Under Pell recursion the Dickson factor D and the hypotenuse z increase and we can speak of the amplitude factor in a complete cycle. Numerical studies for Γ=1, 7 (two branches), 17, 23, 31, 41, 47 and 49 show that the successive values of the amplitude ratio approach 5.83… or the square of $J = \sqrt{2} + 1$ with $JJ^* = 1$. To show the convergence we start with two valid values from a triple, $P_0 = D$ and $P_1 = z$.Then write

$$z = P_1 = (J + \varepsilon)D = (J + \varepsilon)P_0$$

where ε is not necessarily small. After a half cycle

$$P_2 = (2J + 1)P_0 + 2\varepsilon P_0 = \left(\left(2J + JJ^*\right) + 2\varepsilon\right)P_0$$
$$= (J^2 + 2\varepsilon)P_0$$

so that the discrepancy reduces by the relative factor 2/J=0.8284…

After a complete cycle we have

$$P_3 = (J^3 + 5\varepsilon)P_0$$

and we have a reduction $5/J^2 = 0.85786…$

Further coefficients of ε are 12, 29, 70, 169… and we recognise the Pell numbers for our paradigm Sophie triple , starting from the degenerate 0, 1.

Table 2 gives starting values for the first 8 values of ΓWe have shown that the coefficients of ε converge with a semi=amplitude ratio towards J. Thus the convergence is true for all Γ.

Table 12. Starting values of z/D and ε.

Γ	1	7	7	17	23	31	41	47
z/D	5/2	13/4	17/6	25/6	65/24	41/8	89/30	157/60
ε	0.09679	0.83579	0.41491	1.752443	0.284120	2.710787	0.452453	0.252403

Generalised Recursion

The Pell recursion is readily generalised. Write

$$L_{n+2} = aL_n + bL_{n+1}.$$

The semi-amplitude ratio converges towards $G = \frac{1}{2}\left(\sqrt{b^2+4a}+b\right)$. Put $G^* = G - b$ so that $GG^* = a$. For example if a=2 and b=1, then G=2 and the coefficients of the ε tern are 0, 1, 1, 3, 5, 11, 21, 43, 85… and the semi-amplitude ratio converges to 2. We can readily show that if we start $L_0 = a, L_1 = aG$ then $L_n = aG^n$. I look to prove convergence from a closed form as before.

I do not know yet of any use of this generalised recursion.

To address consider the discrepancy coefficients starting with

$$L_0 = 0, L_1 = 1$$

We obtain

$$L_2 = b, L_3 = a + b^2$$
$$L_4 = 2ab + b^2, L_5 = a^2 + 3ab^2 + b^3$$

$L_3 / L_1 = a + b^2$ and $L_5 / L_3 =$

Thos $\left(a^2 + 3ab^2 + b^3\right)/\left(a + b^2\right)$

$$= a + b^2 + \frac{ab^2}{a+b^2}$$

Now

$$G^2 = \tfrac{1}{4}\left(4a + 2b^2 + 2b^2\sqrt{1+\frac{4a}{b^2}}\right);\ 2a + b^2$$

expanding the radical for small a.

Every cycle can be taken to start afresh. Then if a>0 the $\varepsilon-$discrepancy decreases under L-recursion.

A Lemma

Allowing negative sides x,y gives a lemma. Not only is the magnitude of the difference of x and y a member of Γ but so is the magnitude of their sum. Thus the sequence (0,1,1), (3,4,5), (20,21,29), (119,120,169),… yields 1,7,41,239,.. whilst (5,12,13) gives 17,103,… and (8,15,17) gives 27,… all satisfying the mod 8 test. Proof: if x,y,z is a valid triple so are $(\pm x,, \pm y, z)$. But what is a difference in two cases is a sum in the other two. Q.E.D.

This is an interesting but not such a powerful result. It is not the sum but the difference of x, y that is invariant under recursion.

The numbers x, y and z

The extension to negative sides allows us to say what values may be and are taken by the sides x and y in a triple. We know that not only x-y but also x+y lie in Γ and both satisfy the modulo 8 test. Thus the sum (x+y) + (x-y) gives

$$2x(mod 8) = 0 \text{ or } \pm 2$$

If we include the degenerate triple 0,1,1 this tells us that all odd numbers are allowed. But even sides are restricted: $x(\bmod 8) \neq \pm 2$. Thus 2 and 6 are absent.

Just because a number is allowed does not prove it present but we may easily show this to be true. From Dickson for odd sides with $\delta_o = 1$ we have $x = D + \Delta_o = 2\delta + 1$ and thus, allowing the degenerate case, all odd numbers appear along the vertical axis.

For even values put $\delta = 1$ so that $x = D + \Delta_e = 2\delta_o + 2$. Then along the horizontal axis the values are $x = 2\delta_o + 2 = 4n$ successively 4,8,12,16,… all the numbers satisfying $2z(\bmod 8) = 0$. All permitted sides z and y are present. There are no barred values such as 2, 6, 10 etc.

We can get a similar result from the basic relation $x^2 + y^2 = z^2$. If y is odd then y(mod8)=z(mod8)=1 and x(mld8)=0.

We see that the sets x and y are full: all allowed values are present. This is not the case or the set of z-values. The figure shows z-values near the origin of the Dickson quadrant.

Table 13. Hypotenuse values

4	41	65		
3	25	X	73	
2	13	29	53	
1	5	17	37	65
$\psi = (\delta_o, \delta)$	**1**	**3**	**5**	**7**

We add 1 to these values from the degenerate triangle 0,1,1. We have $z = D + \Delta_o + \Delta_e = 2\delta_o\delta + \delta_o^2 + 2\delta^2$ so that as we go outwards z increases. Thus 3, 7, 9 and 11 are absent from the set of hypotenuse.

It helps to note that from Dickson we have $z = 2\delta_o\delta + 2\delta^2 + \delta_o^2 = (\delta + \delta_o)^2 + (\delta)^2$. Thus $z = m^2 + n^2$ where m and n are integers of opposite parity, odd and even. How curious that this has the same form as our original triple $z^2 = x^2 + y^2$ Thus we have $z(\bmod 8) = m^2(\bmod 8) + n^2(\bmod 8) = 0/4 + 1 = 1 \text{ or } 5$.

Table 14. z-values $z = (\delta_o + \delta)^2 + \delta^2$

20	841	929	X	1129	1241	11361	1489	X	1769	1921
19	761	845	9337	1037	1145	1261	1385	1517	16557	1805
18	685	X	853	949	X	1165	1285	X	1549	1693
17	513	691	653	847	965	1073	1189	1313	1445	1585
16	545	417	497	785	991	985	1097	1217	1345	1481
15	481	X	X	709	X	901	1009	X	1249	1381
14	421	485	557	X	725	821	925	1037	1157	1285
13	365	425	493	569	653	745	X	973	1069	1193
12	313	X	433	505	X	673	769	X	965	1105
11	263	315	377	445	521	X	697	797	905	1021
10	221	269	X	389	461	5541	629	X	829	941
9	181	X	277	3337	X	481	565	X	757	865
8	145	18511	233	289	3353	425	475	593	589	793
7	113	149	193	X	305	373	449	533	625	725
6	85	X	157	205	X	325	397	X	565	661
5	61	89	X	169	221	281	349	X	509	601
4	41	65	97	137	185	241	305	377	475	545
3	25	X	73	109	X	205	265	X	409	493
2	13	29	53	85	125	173	229	293	365	445
1	5	17	37	65	101	145	197	257	325	401
$\psi = (\delta_o, \delta)$	1	3	5	7	9	11	13	15	17	19

We note occasional duplications such as 65 at 7,1 and 3,4.

z-lacunae

The following values are missing from the quadrant:

Mod8=1: 9, 33, 49, 57,81 ,89, …

Mod8=5: 21, 45, 69,7783, ...

These values satisfy the test modulo 8 but cannot be written in the form $z = (\delta_o + \delta)^2 + \delta^2$.

Thus 9 cannot be represented as $m^2 + n^2$ without using m=0 which in turn gives a reducible triple 0,9,9.

In the range z<900; there are some 225 valid values but only some 150 cells of which say 20 are reducible or duplicates. Thus z is some 40 % sparse.

So we see that whereas x and y are limited by the test modulo 8 and all admissible values are present, this test is insufficient for the hypotenuse z. 9 satisfies the test but cannot be written $m_2 + n_2$. Rather we should employ the constructive proof that $z = m^2 + n^2$ where m and n are integers of alternate parity, odd and even.

I have recently been introduced to the work of Vella, Vella and Wolf (henceforth VVW).

For example, in the triple 120, 121, 169 we have 169 a square number. This can only be so if that number is itself an hypotenuse: 5, 12, 13.

Can z be the hypotenuse of more than one triple? This calls for more than one pair $z = m^2 + n^2$. VVW show that this turns on the number of prime factors k of z and is given by 2^{k-1}. Thus z=5 is unique, ignoring negative sides. $65 = 5 \times 13$ has two triples 16,63,65 and 3,56,65. Similarly $z = 145 = 5 \times 29$ has two triples; can you find them? But $z = 169 = 13^2$ has only one prime factor and occurs uniquely at 120,121,169. And this factor 13 itself appears uniquely in 5,112,13.

Then $z = 3485 = 5 \times 17 \times 41$ has four triples: m n=59,2; 58,11; 53, 26; 46, 37 and nowhere else. Again, readers might like to find and check another case.

The Lewins Conjecture

We recollect the values found for $\Gamma = |\Lambda = \Delta_o - \Delta_e|$ Table x.

Table 15. Λ values as a function of δ / δ_o

20	799	791	X	751	7719	679	631	X	511	439
19	721	713	697	673	641	601	553	497	433	**X**
18	647	X	623	599	X	527	479	X	359	**2287**
17	577	569	553	529	497	457	409	353	**X**	**217**
16	511	503	487	463	431	391	341	287	**223**	**151**
15	449	X	X	401	X	329	281	X	**161**	**89**
14	391	3183	367	X	311	271	223	167	**103**	**31**
13	337	329	313	289	257	217	X	112	**49**	**-23**
12	287	X	263	239	X	167	**119**	X	**-1**	**-73**
11	241	233	217	193	161	X	**73**	**17**	**-47**	**-119**
10	199	191	X	151	119	**79**	**31**	**X**	**-89**	**-161**
9	161	X	137	113	X	**41**	**-7**	**X**	**-127**	-199
8	127	119	103	**79**	**47**	**7**	**-41**	**-97**	161 7×23	-122
7	+8	89	73	**X**	**17**	**-23**	23-**71**	-127	-191	-263
6	71	X	**47**	**23**	**X**	**-49**	-97	X	-127	-289 17^2
5	49	41	**X**	**-1**	**-31**	-71	hotvella	X	-239	-311
4	31	23	**7**	**-17**	-49 -7^2	-89	-137	-193	-2557	-329 7×47
3	17	X	**-7**	-31	X	-103	-151	X	-271	-343 7^3
2	7	**-1**	-17	-41	-73	-113	-161 -7×23	-217 -7×31	-281	-353
1	1	-7	-23	-47	-79	-119 -7×17	-167	-223	-284 7×41	-359
$\psi =$ (δ, δ_o)	**1**	**3**	**5**	**7**	**9**	**11**	**13**	**15**	**17**	**19**

X co-factor Bold: recursor

We see that in general there are starting values in the upper and lower segments $\pm\Lambda$ followed by twin traces in the central wedge.

Exceptions include $\Lambda = 1$ followed by a single trace and, for example, 17 and $119 = 17 \times 7$ which have double starts.

We see that in the range studied, the Γ – values are limited to those primes p (all odd) satisfying $p(\bmod 8) = \pm 1$ or products of such primes.

Our conjecture then is in two parts:

1. The set Γ is limited to prime factors and their products satisfying $p(\bmod 8) = \pm 1$. Thus 7x7=49 is valid but not 3x3 even though 9(mod8))=1. Since all products of valid primes satisfy the test we have $\Gamma(\bmod 8) = \pm 1$ if the conjecture is valid.

2. That every possible product of valid primes can indeed be found in the set Γ. That is, we have a set Λ

$$\Lambda = \pm \prod_{n=1, ,=1}^{\infty,\infty} p_n^m$$

where p satisfies modulo 8=± 1 and consequently $\Gamma(\bmod 8) = \pm 1$.

I can offer a proof for the first part as follows. Our equation is

$$\Lambda = \Delta_e - \Delta_o = 2\delta^2 - \delta_o^2$$

where δ, δ_o are co-prime integers and δ_o is odd.

Such an equation is the proper subject of Gauss's quadratic reciprocity theorem, a deep theorem at the heart of formal algebra.[5] Witness to this complexity is the claim that there are over 200 proofs of the theorem, including one found on Gauss at his death. Fortunately there is an easier proof.

The odd series $\Delta_o = 1, 9, 25, ...$ has a general term $(2n[1)^2$ and hence a stepwise increment $(2n+1)^2 - (2n-1)^2 = 8n$. Thus $\Delta_o(\bmod 8) = 1$. For $\Delta_e(\bmod 8) = 2\delta^2(\bmod 8)$ the odd integers therefore contribute 2 and the even give $2(2n)^2 = 8n^2$ and contribute zero. Thus

$$\Gamma(\bmod 8) = (2 \text{ or } 0)\text{-}1 = \pm 1$$

We have proved that all primes p in the set are limited to $p(\bmod 8) = \pm 1$.We still must show that such an invalid prime cannot be a factor of any member of the set even though, for example, 0^2 satisfies the modulo 8 test.

To do this, we turn to the q-values or residuals modulo p. At any multiple of p the residual is zero but this leads to a false, reducible

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[5] See for example http://www.math.uga.edu/~pete/thuelemmav3.pdf

solution. Between successive false solutions the even index has p-1 values but from the symmetry only half can be distinct: $n^{2p-n2}(mlsp) = (p-m)^2 (\mathrm{mod}\, p)$ leaving (p-1)/2 for the odd-index. Thus the two fields can indeed be disjoint. Thus for an invalid p=5 the full series gives the field 2,3,3,2 and the odd series gives 1,4. For p=7, valid, we have 2,1,4,4,1,2 and 1,2,4, the first disjoint and the second joint.

But we have shown that if p is an invalid prime it cannot appear in Γ so that the two fields are indeed disjoint and hence we have proved that if $p(\mathrm{mod}\, 8) \neq \pm 1$ it cannot appear as a factor in the set.

We have checked the harmonics up to $\Gamma = 49$. All are present.

Table 14. Harmonics up to R=2500[6]

Γ	Factors	$\psi = (\delta_o, \delta_e)$	Γ	Factors	$\psi = (\delta_o, \delta_e)$	Γ	Factors	$\psi = (\delta_o, \delta_e)$
1	1	1,1	17	17	5,2	23	23	5,1
7	7	1,2	119	17×7	19,6	391	23×17	21,5
49	7^2	9,4	833	17×7^2	33,2	161	23×7	12,2
343	7^3	19,3	289	17^2	19,6	1127	23×7^2	37,11
2401	7^4	51,10	2023	$17^2 \times 7$	45,1	529	23^2	27,10
31	31	7,3	41	41	,2	47	47	477,1
713	31×23	29,2	1271	41×31	27,7	1927	47×41	45,1
527	31×17	12,1	943	41×23	31,3	1457	47×31	53,26
217	31×7	25,2	697	41×17	27,4	1081	47×23	33,2
1519	31×7^2	39,1	287	41×7	17,1	799	47×17	31,9
961	31^2	33,8	2009	41×7^2	47,10	329	47×7	23,10
			1681	41^2	59,30	2303	47×7^2	53,5
						2209	47^2	51,4

[6] For prime numbers: http://primes.utm.edu/lists/small/100000.txt

Γ	Factors	$\psi =$ (δ_o, δ_e)	Γ	Factors	$\psi =$ (δ_o, δ_e)	Γ	Factors	$\psi =$ (δ_o, δ_e)
71	71	11,5	73	73	9,2	79	79	9,1
2201	71×31	49,10	2263	73×31	61,27	2449	79×31	57,29
1633	71×23	51,22	1679	73×23	41,1	1817	79×23	45,4
1207	71×17	35,3	1241	73×17	37,8	1343	79×17	55,22 9
497	71×7	23,4	511	73×7	23,3	553	79×7	25,6
89	89	11,4	97	97	15,8	103	103	11,3
2047	89×23	47,9	2231	97×23	53,17	2369	103×23	49,4
1513	89×17	39,2	1649	97×17	41,4	1751	103×17	43,7
623	89×7	25,1	679	97×7	27,5	721	103×7	27,2
113	113	11,2	127	127	15,7	137	137	13,4
1921	113×17	47,12	2159	127×17	47.5	2329	137×17	49,6
791	113×7	29,5	889	127×7	31,6	959	137×7	2,1
151	151	13,3	167	167	13,1	191	191	17,7
1057	151×7	33,4	1169	167×7	37,10	1337	191×7	37,4
199	199	19,9	223	223	15,1	233	233	25,14
1393	199×7	39.9	1561	223×7	43,12	1631	233×7	41,5
241	241	32,10	257	257	35,22	263	263	19,7
1687	241×7	43,9	1799	257×7	43,5	1841	263×7	43,2
271	271	17,3	281	281	17,2	311	311	19,5
1897	271×7	45,8	1967	281×7	47,11	2177	311×7	53,4
313	313	21,8	337	337	25,12	353	353	21,2
2191	313×7	47,3	2359	337×7	51,11	2471	353×7	53,13

Γ	ψ	Γ	ψ	ψ Γ	ψ	Γ	ψ	Γ	ψ
359	19,1	367	23,9	383	25,11	401	23,8	4009	21,4
431	23,7	433	21,2	439	21,1	449	31,16	457	23,6
463	25,9	479	23,5	487	27,11	503	3+,13	521	23,2
569	31,14	577	33,16	599	29,11	601	27,8	607	25,3
617	25,2	631	27,7	641	29,10	647	35,17	673	31,2
719	31,11	727	27,1	743	29,7	751	33,13	761	31,10
7669	29,6	809	294	823	29,3	839	29,1	857	37,16
863	31,7	881	31,2	887	35,13	911	31,5	919	37,15
929	31,4	937	35,12	953	31,2	967	43,41	977	37,14
983	37,11	991	33,7	1009	39,16	1031	37,13	1033	41,18
1039	33,25	1049	43,20	1063	35,9	1087	33,1	1097	35,8
1103	43,17	1129	39,14	1153	35,6	1193	35,4	1201	37,7
1217	35,2	1223	35,1	1231	41,15	1249	51,26	1279	39,11
1327	47,21	1361	37,2	1367	37,1	1399	43,15	1409	47,20
1423	39,7	1433	49,22	1439	49,11	11447	45,17	1471	39,5
1481	41,10	1487	47,19	1489	39,4	15543	51,23	1553	41,8
1559	51,25	1567	55,27	11601	49,20	1607	43.11	1609	41,6
1657	53,24	1663	44,3	1697	47,16	1721	43,8	1753	49,18
1759	47,15	1777	45,6	1783	45,11	1801	51,20	1823	49,17
1831	43,3	1817	43,4	1871	47,13	1879	51,19	1889	49,16
1913	59,28	1951	49,15	1993	45,4	1999	57,25	2017	45,2
2039	361,29	2063	449,13	2081	47,8	2087	25,8	2089	51,16
2111	47,7	2113	49,12	2129	59,26	2137	47,6	2143	55,21
2153	61,28	2161	53,118	2207	47,1	2239	47,9	2273	49,8
2281	57,22	2287	63,29	2297	53,16	2311	67,33	2351	49,5
2377	55,18	2383	49,3	2393	49,2	2399	49,1	2417	53,14
2423	59,23	2441	67,32	2447	55,17	2473	51,8		*e.o.e*

Thus the conjecture is valid up to R=2500.

We have no rigorous proof for the second part of our conjecture but we see it supported as far as the table goes. A constructive proof would be ideal but an attempt to show every number satisfying the modulo 8 test will fail; 15 has no solution.

A computer program to extend the range might go as follows:

Select the range R, say one million;

Find all valid primes p by the usual sieve algorithm up to $\sqrt{R}$, say 1000;

Extend the table $\psi = (\delta, \delta_o)$ **to** $\delta_o^2 = R$**;**

Construct all product of the primes p up to R;

Check for the occurrence of each harmonic between vertical columns in the lower wedge from $\delta_o = m$ where m is odd and just satisfies $m^2 > \Gamma$ to $\delta_o = n$ where b odd just satisfies $(n-1)^2 > 2 + \Gamma$.

I am delighted that Alister Perrott has written such as programme checking the occurrence of x+y, which can be shown to be an equivalent set. He had validated my hand results to 2500 and extended them to 5,000,000. The programme runs on a PC and took some 10 hours to reach 5,000,000. He has kindly made the programme availabled at:

http://www.shelaghlewins.com/other_stuff/triangles_perl_script.txt

Conclusions

Our journey through Pythagorean triples has seen rich mathematics and famous mathematicians. *Oft have I travelled in the realms of gold.* We met Euclid, Euler, Gauss, Pascal, Dickson and above all, Pythagoras on the way. We have shown the equivalence of the Dickson and Pell sequences , the invariance of Γ and the convergence to the silver ratio $J = \sqrt{2} + 1$. We have achieved a Fourier decomposition of the set of those numbers representing the short sides of our triples into prime factors satisfying the modulo 8 test.

Fancifully, eight makes me think of the musical scale. If indeed all possible fundamental primes, their overtones and harmonics are present in the set Γ, may we say that it rings with the music of the spheres?

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Vella, Dominic, Vella, Alfred and Wolf, Julia, *An Extension of the Fundamental Theorem on Right-Angled Triangles,* see http://www.damtp.cam.ac.uk/user/dv211/mathgaz05.pdf and http://people.maths.ox.ac.uk/vella.

The English mathematician John Pell lived from 1611 to 1685. A fuller account of Pell numbers and the associated Pell-Lucas numbers may be found at:

http://mathworld.wolfram.com/PellNumber.html

A proof of Euler's formula can be found at:

http://mathworld.wolfram.com/SquareTriangularNumber.html